

A Magnetic Field Diffusion Problem

L. T. KLAUDER, JR.

Research Laboratories, General Motors Corporation, Warren, Michigan 48090

(Received 11 March 1968)

This paper presents a solution to the problem of determining the lift and drag on a long, straight, current-carrying wire moving side-ways over a conducting semi-infinite slab.

People who have thought about futuristic forms of ground transportation have sometimes suggested¹ that very high speed vehicles might be supported by magnetic fields. This suggestion tends to arise in connection with speeds at which wheels cannot be used and at which aerodynamic effects would be troublesome if the guideway tunnel were not evacuated.

When the author encountered the foregoing suggestion he was curious as to whether there might be any scheme for realizing magnetic support and, if so, what its power consumption would be. It appeared that support based on interaction between independent magnetic fields would be inherently unstable, so attention was directed to support based on the interaction of a moving magnet and a flat electrically conducting slab. The magnetic field penetrates the conducting slab by diffusion which, if the conductivity is high, can be slow relative to the motion of the magnet. If the diffusion is "slow", the magnetic field is compressed between the magnet and the slab and exerts lifting force on the magnet. However, the field still penetrates the slab to some extent, and a force must be applied to drive forward through the slab that portion which does penetrate. Thus, there will be drag as well as lift, and one would like to find their ratio.

With this question in mind, the author considered a boundary value problem that simulates the mechanism described above. The solution is fairly simple and, since most present-day textbooks on electricity and magnetism have little to say about magnetic field diffusion, the solution may be of interest to lecturers.

We consider a straight, current-carrying wire moving over a slab of conducting material. In order to make the problem simple, we consider the wire to be infinitely long. For further simplicity we consider the slab to be infinitely thick. The arrangement and symbols are shown in Fig. 1. The object is to determine the lift force

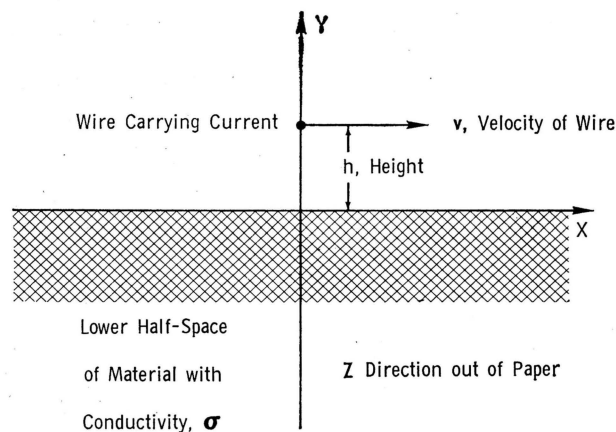


FIG. 1. Wire moving horizontally above conducting slab. Axis of wire is along z direction.

per unit length, F_y , and the drag force per unit length, F_x , acting on the moving wire. These

¹ *Survey of Technology for High Speed Ground Transport* prepared for the U.S. Department of Commerce by the Massachusetts Institute of Technology, June 1965, Part 1, p. V-180.

forces are exerted on the wire by the currents induced in the slab.

The equation governing the diffusion of the magnetic field in a conducting, nonmagnetic (i.e., $\mu=1$) medium is, in Gaussian units,

$$\nabla^2 \mathbf{B} = (4\pi\sigma/c^2) (\partial \mathbf{B} / \partial t).$$

In our arrangement $\mathbf{B}$ is independent of z and depends on x and t only through the combination $x-vt$, i.e.,

$$\mathbf{B}(x, y, t) = \mathbf{B}(x-vt, y, 0).$$

Thus in our case the equation may be written:

$$(\partial^2 \mathbf{B} / \partial x^2) + \lambda (\partial \mathbf{B} / \partial x) + (\partial^2 \mathbf{B} / \partial y^2) = 0,$$

where

$$\lambda \equiv 4\pi\sigma v / c^2.$$

We first seek a solution whose x dependence is as e^{ikx} . Such a solution must satisfy

$$(\partial^2 \mathbf{B}_k / \partial y^2) (x, y) = (k^2 - ik\lambda) \mathbf{B}_k(x, y),$$

and must therefore be proportional to

$$\exp(ikx + ly),$$

where

$$l \equiv \pm (k^2 - ik\lambda)^{1/2}.$$

We are seeking a solution which goes to zero as y goes to $-\infty$; therefore the sign of l is to be chosen so that $\text{Re } l > 0$. Any boundary condition at the surface may be met from these solutions by Fourier composition. Thus we may write the solution within the slab ($t=0$) as

$$\mathbf{B}(x, y) = \int_{-\infty}^{\infty} dk \mathbf{b}(k) \exp(ikx + ly).$$

Using $\nabla \times \mathbf{B} = (4\pi/c) \mathbf{j}$ we find the current density induced in the conducting slab to be

$$j_z = \frac{c}{4\pi} \int_{-\infty}^{\infty} dk [ikb_y(k) - lb_x(k)] \exp(ikx + ly).$$

The magnetic field to which this gives rise is given (in or out of the slab) by

$$B_x^j(x, y) = \frac{-2}{c} \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \frac{j_z(x', y')(y-y')}{(x-x')^2 + (y-y')^2}.$$

Integrating we find for $y \geq 0$

$$B_x^j(x, y \geq 0) = -\frac{1}{2} \int_{-\infty}^{\infty} dk [ikb_y(k) - lb_x(k)] \times \frac{\exp(ikx - |k|y)}{l + |k|}.$$

Analogously we find

$$B_y^j(x, y \geq 0) = -\frac{1}{2}i \int_{-\infty}^{\infty} dk [ikb_y(k) - lb_x(k)] \times \text{sgn } k \frac{\exp(ikx - |k|y)}{l + |k|},$$

where

$$\text{sgn } k \equiv k / |k|.$$

Furthermore, the field due to the current in the wire is

$$B_x^w(x, y) = \frac{2I}{c} \frac{y-h}{x^2 + (y-h)^2},$$

$$B_y^w(x, y) = \frac{-2I}{c} \frac{x}{x^2 + (y-h)^2}.$$

We now have two different expressions for the magnetic field at the surface of the conducting slab. Equating the expressions for $B_x(x, 0)$ we have

$$\int_{-\infty}^{\infty} dk b_x(k) e^{ikx} = \frac{-2Ih/c}{x^2 + h^2} - \frac{1}{2} \int_{-\infty}^{\infty} dk \times [ikb_y(k) - lb_x(k)] \frac{e^{ikx}}{l + |k|}.$$

Taking the Fourier transform of this equation (i.e., multiplying by $(1/2\pi) \exp(-ik'x)$ and integrating over all x) we obtain the equation

$$b_x(k) = \frac{-I}{c} \exp(-h|k|) - \frac{ikb_y(k) - lb_x(k)}{2l + 2|k|}.$$

Equating the two expressions for $B_y(x, 0)$ and taking the Fourier transform we obtain analogously,

$$\text{sgn } k b_y(k) = \frac{iI}{c} \exp(-h|k|) - \frac{i[ikb_y(k) - lb_x(k)]}{2l + 2|k|}.$$

Solving for $b_x(k)$ and $b_y(k)$ we obtain

$$b_x(k) = -2I \exp(-h |k|) l/c(l + |k|),$$

and

$$b_y(k) = 2iI \exp(-h |k|) k/c(l + |k|).$$

Then for the expression which occurs in the formula for the induced current density we have

$$ikb_y(k) - lb_x(k) = 2I \exp(-h |k|) (l - |k|)/c.$$

We are now in a position to compute the x and y components of the force on the wire.

$$F_x = \frac{+2I}{c^2} \int_{-\infty}^{\infty} dx' \int_{-\infty}^0 dy' \frac{j_z(x', y') x'}{x'^2 + (h - y')^2},$$

which upon substituting and integrating gives

$$F_x = \frac{+I^2 i}{c^2} \int_0^{\infty} dk \exp(-2hk) \times \left[\frac{l(k) - k}{l(k) + k} - \frac{l(-k) - k}{l(-k) + k} \right]. \quad (1)$$

Similarly,

$$F_y = \frac{I^2}{c^2} \int_0^{\infty} dk \exp(-2hk) \times \left[\frac{l(k) - k}{l(k) + k} + \frac{l(-k) - k}{l(-k) + k} \right]. \quad (2)$$

where the quantity $l(k)$ is given in terms of k by the formula

$$l(k) = (1/\sqrt{2}) \left[k(k^2 + \lambda^2)^{1/2} + k^2 \right]^{1/2} + (i \operatorname{sgn} k / \sqrt{2}) \left[k(k^2 + \lambda^2)^{1/2} - k^2 \right]^{1/2}.$$

Although the above Laplace transforms are complicated, there is an approximate treatment which is satisfactory for the purpose of estimating F_y/F_x when the velocity v is large. This approximation depends on the relative sizes of h and λ .

For copper $\sigma = 5.3 \times 10^{17}$ /sec (gaussian units); let us take $v = 500$ mph $= 2.2 \times 10^4$ cm/sec. Then

$\lambda = 4\pi\sigma v/c^2 = 160$ /cm. The factors (here $\bar{l}$ stands for $l(-k)$)

$$+i \left[\frac{l-k}{l+k} - \frac{\bar{l}-k}{\bar{l}+k} \right] = \frac{-2[2k(k^2 + \lambda^2)^{1/2} - 2k^2]^{1/2}}{k + [2k(k^2 + \lambda^2)^{1/2} + 2k^2]^{1/2} + (k^2 + \lambda^2)^{1/2}}, \quad (3)$$

and

$$\left[\frac{l-k}{l+k} + \frac{\bar{l}-k}{\bar{l}+k} \right] = \frac{2[(k^2 + \lambda^2)^{1/2} - k]}{k + [2k(k^2 + \lambda^2)^{1/2} + 2k^2]^{1/2} + (k^2 + \lambda^2)^{1/2}}, \quad (4)$$

are seen to be of order unity or less for all k . Then taking $h = 1$ cm, one sees that for those values of k which give any significant contribution to the integrals in (1) and (2), we may expand (3) and (4) in powers of k/λ . Doing this and carrying out the integrals we obtain

$$F_x = \frac{-I^2}{c^2 h} \left[\frac{1}{2} \left(\frac{\pi}{h\lambda} \right)^{1/2} \right] \left(1 - \frac{2}{(\pi h\lambda)^{1/2}} + \frac{3}{8h\lambda} + \dots \right),$$

$$F_y = \frac{I^2}{c^2 h} \left(1 - \frac{\pi^{1/2}}{2(h\lambda)^{1/2}} + 0 + \dots \right).$$

For the case which we are considering (i.e., σ of copper, $v = 500$ mph, and $h = 1$ cm) $(h\lambda)^{-1/2} = 0.079$ so that for approximate results we may keep just the first term of each series. This gives for the ratio of drag to lift

$$|F_x|/F_y = \frac{1}{2} (\pi/h\lambda)^{1/2} = c/4(h\sigma v)^{1/2},$$

which in our case is 0.07.

Note Added in Proof: J. R. Powell and G. R. Danby, A.S.M.E. Paper 66-WA/RR-5, have proposed a magnetic vehicle support scheme which would give better performance than the scheme discussed above.